A treatment of the quantum partial entropies in the atom-field interaction with a class of Schrödinger cat states

A.-S. F. Obada[1], H. A. Hessian[2] and Mahmoud Abdel-Aty[3,4]

[1]Mathematics Department, Faculty of Science Al-Azhar University, Nasr City, Cairo, Egypt

[2]Institut für Theoretische Physik, Universität Innsbruck, Technikerstrasse 25, A-6020

Innsbruck, Austria

[3]Mathematics Department, Faculty of Science, South Valley University, 82524 Sohag, Egypt

Abstract: This communication is an enquiry into the circumstances under which entropy and subentropy methods can give an answer to the question of quantum entanglement in the composite state. Using a general quantum dynamical system we obtain the analytical solution when the atom initially starts from its excited state and the field in different initial states. Different features of the entanglement are investigated when the field is initially assumed to be in a coherent state, an even coherent state (Schrodinger cate state) and a statistical mixture of coherent states. Our results show that the setting of the initial state and the Stark shift play important role in the evolution of the sub-entropies and entanglement.

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1   Introduction

The quantum entropy and entanglement is a vital feature of quantum information. It has important applications for quantum communication and quantum computation, for example, quantum teleportation, massive parallelism of quantum computation and quantum cryptographic schemes [1-3]. Therefore, it is very essential and interesting how to measure the entanglement of quantum states. Recently much attention has been focused on the entanglement of the field and atom when the system starts from a pure state [4-16]. Also, in the context of the initial mixed state some studies have been reported [17-20]. In this context it was shown that calculating the partial entropies of the field or the atom can

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[4]Corresponding author: abdelatyquantum@yahoo.co.uk

be used as an operational measure of the entanglement degree of the generated quantum state. One finds that the higher the entropy, the greater the entanglement. Starting from an initial atom-field product state one can find perfectly entangled states between field and atom at certain later times even for initial coherent states with large photon number [4-8]. However, the time evolution of the field (atomic) entropy reflects the time evolution of the degree of entanglement only if one deals with a pure state of the system with zero total entropy.

The quest for proper entanglement measures has received much attention in recent years [1,9]. From the identification and study of properties of such measures a gain of insight into the nature of entanglement is expected. In turn, their computation for particular states provide us with an account of the resources present in those states. Because one needs to understand the best way to benchmark states for quantum information protocols, here we examine the quantum partial entropies in the atom-field interaction for more general entangled states. From a practical point of view, an implementation of the quantum entropy will be used to measure the entanglement degree when the atom is assumed to be in its excited state and the field initially is in a coherent state, superposition state and a statistical mixture of two coherent states. As far as we are aware in the previous investigations, that have dealt with the present problem, the initial system density matrix is taken to be a product of two states of the factored form. The atom is often taken to be in the excited pure state or mixed state and the radiation field is taken to being a pure state density matrix. So, the present task is a nontrivial issue, since we look at the mixed state entanglement from other direction taking into account the entropy of the field not equal to the entropy of the atom, in this case. To overcome such a difficulty, we employ the quantum von-Neumann entropy to measure the entropy of the atom while a numerical method will be used to calculate the quantum entropy of the field.

The material of this paper is arranged as follows. In section 2, we find the exact solution of the system and write the expressions for the final state vector at any time $t > 0$. We investigate the quantum field (atomic) entropy and the atom-field entanglement in section 3. Finally, numerical results and conclusions are provided in section 4.

2   The model

The system we will consider here consists of a two-level atom interacting with a single-mode quantized field via k-quanta processes. The Hamiltonian in the rotating wave

approximation [8,20], can be written as ($\hbar = 1$):

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{in}, \tag{1}$$

where

$$
\begin{aligned}
\hat{H}_F &= \omega \hat{a}^\dagger \hat{a}, \\
\hat{H}_A &= \frac{\omega_\circ}{2} \hat{\sigma}_z, \\
\hat{H}_{in} &= \hat{a}^\dagger \hat{a}(\beta_1 |g\rangle\langle g| + \beta_2 |e\rangle\langle e|) + \lambda(\hat{a}^{\dagger k}\hat{\sigma}_- + \hat{a}^k \hat{\sigma}_+),
\end{aligned} \tag{2}
$$

where ω is the field frequency and $\omega_\circ$ is the transition frequency between the excited and ground states of the atom. We denote by $\hat{a}$ and $\hat{a}^\dagger$ the annihilation and the creation operators of the cavity field respectively. β_1 and β_2 are parameters describing the intensity-dependent Stark shifts of the two levels that are due to the virtual transition to the intermediate relay level, λ is the effective coupling constant, $\hat{\sigma}_z$ is the population inversion operator, and $\hat{\sigma}_\pm$ are the "spin flip" operators, with the detuning parameter $\Delta = \omega_\circ - k\omega$.

Let us consider, the atom starting in its excited state $|e\rangle$, i. e.,

$$\rho_0^a = |e\rangle\langle e|, \tag{3}$$

and we are going to assume that the initial single mode electromagnetic field inside the cavity is in a superposition state of the kind:

$$\rho_0^f = \frac{1}{A}\left(|\alpha\rangle\langle\alpha| + r^2| - \alpha\rangle\langle -\alpha| + r|\alpha\rangle\langle -\alpha| + r| - \alpha\rangle\langle\alpha|\right), \tag{4}$$

where $A = (1 + r^2 + 2r\exp(-2\alpha^2))$, with α real. The parameter r takes the values -1, 0 and 1, which corresponds to an odd coherent state, a coherent state and an even coherent state, respectively. While for certain classes of states a superpositions of coherent states, methods solely based on linear optical elements like beam splitters and photodetections could be found [21], an implementation covering other classes of entangled states remains a challenge.

Also we want to see how different would the behavior of the system be if the input state is a statistical mixture of states $|\alpha\rangle$ and $| - \alpha\rangle$, i.e.,

$$\rho_0^f = \frac{1}{2}\left(|\alpha\rangle\langle\alpha| + | - \alpha\rangle\langle -\alpha|\right). \tag{5}$$

It is to be noted that when we put $r = 0$ in equation (4) we get the same result as in Ref. 13. It is expedient to expand the atom-field state in terms of the dressed states:

$$
\begin{aligned}
|\Psi_+^{(n)}\rangle &= \sin\theta_n |n, e\rangle + \cos\theta_n |n + k, g\rangle, \\
|\Psi_-^{(n)}\rangle &= \cos\theta_n |n, e\rangle - \sin\theta_n |n + k, g\rangle,
\end{aligned} \tag{6}
$$

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which are the eigenstates of the interaction Hamiltonian, where

$$
\begin{aligned}
\hat{H}|s, g\rangle &= E_0|s, g\rangle, \quad 0 \le s < k \\
\hat{H}|\Psi_\pm^{(n)}\rangle &= E_\pm^{(n)}|\Psi_\pm^{(n)}\rangle,
\end{aligned}
\tag{7}
$$

with the eigenvalues E_0 and $E_\pm^{(n)}$

$$
\begin{aligned}
E_\pm^{(n)} &= \omega\left(n + \frac{k}{2}\right) + \frac{\omega_0}{2} + +\frac{1}{2}[n\beta_2 + \beta_1(n+k)] \pm \mu_n, \\
E_0 &= \left(s\beta_1 - \frac{\Delta}{2}\right),
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\mu_n &= \sqrt{\nu_n^2 + \tau_n^2}, \\
\nu_n &= \frac{\Delta}{2} + \frac{1}{2}(\beta_2 n - \beta_1(n+k)), \\
\tau_n &= \lambda\sqrt{\frac{(n+k)!}{n!}}.
\end{aligned}
\tag{9}
$$

μ_n is a modified Rabi frequency. The angle θ_n is given by

$$
\theta_n = \sin^{-1}\left(\frac{\tau_n}{\sqrt{(\nu_n - \mu_n)^2 + \tau_n^2}}\right).
\tag{10}
$$

The unitary operator $\hat{U}_t$ can be written as

$$
\begin{aligned}
\hat{U}_t &= \sum_{n=0}^{\infty}\left\{\exp(-itE_+^{(n)})|\Psi_+^{(n)}\rangle\langle\Psi_+^{(n)}| + \exp(-itE_-^{(n)})|\Psi_-^{(n)}\rangle\langle\Psi_-^{(n)}|\right\} \\
&+ \sum_{s=0}^{k-1}\exp(-itE_0)|s, g\rangle\langle g, s|.
\end{aligned}
\tag{11}
$$

Despite being straightforwardly solvable in this way, the JC-model is well-known for the fact that the time-evolution of most expectation values is usually expressible only in series form. Having obtained the explicit forms of the unitary operator $\hat{U}_t$, for the system under consideration then the eigenvalues and the eigenfunctions can be used to discuss many features concerning the field or the atom.

Bearing these facts in mind we find that the evolution operator $\hat{U}_t$ takes the next from

$$
\rho_t = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix},
$$

where $(\rho_i)_{nm} = \langle n|\rho_i|m\rangle$, $i = 1, 2, 3, 4$. $(\rho_1)_{nm} = A_n(t)A_m^*(t)$, $(\rho_2)_{nm} = A_n(t)B_{m-k}^*(t)$, $(\rho_3)_{nm} = B_{n-k}(t)A_m^*(t)$, and $(\rho_4)_{nm} = B_{n-k}(t)B_{m-k}^*(t)$. The coefficients $A_n(t)$ and $B_n(t)$ are given by

$$
\begin{aligned}
A_n(t) &= q_n C_n \exp[-i\lambda t \delta_+(n)] \left(\cos \lambda t \mu_n - i\eta_n \frac{\sin \lambda t \mu_n}{\mu_n} \right), \\
B_n(t) &= -iq_n C_n \nu_n \exp[-i\lambda t \delta_+(n)] \frac{\sin \lambda t \mu_n}{\mu_n}, \\
R^2 &= \sqrt{\beta_1/\beta_2}, \quad \eta_n = \frac{\delta}{2} + \delta_-(n), \quad \delta = \frac{\Delta}{\lambda}, \\
\delta_\pm(n) &= \begin{cases} \frac{1}{2R}[n \pm R^2(n+k)], & when \ R \neq 0 \\ 0 & when \ \beta_i = 0, \end{cases}
\end{aligned}
\tag{12}
$$

where

$$
C_n = [\frac{1}{\sqrt{A}}(1 + r(-1)^n)],
$$

for the initial condition (3), while for the initial condition (4) is

$$
C_n = [\frac{1}{\sqrt{2}}(\delta_i + (-1)^n \delta_j)],
\tag{13}
$$

with δ_i, δ_j satisfying the two following condition, (a) $\delta_i = \delta_j = (\delta_i)^2 = (\delta_j)^2 = 1$, and (b) $\delta_i.\delta_j = 0$,

$$
|\alpha\rangle = \sum_{n=0}^\infty q_n|n\rangle = \sum_{n=0}^\infty e^{-\alpha^2/2} \frac{\alpha^n}{\sqrt{n!}}|n\rangle.
\tag{14}
$$

With the final state obtained, any property related to the atom or the field can be calculated. Employing the reduced density operator for the atom or the field, we investigate the properties of the entropies (S_a, S_f) and hence entanglement.

3 Entropy and subentropy

There is growing interest in the roles of nonadditive measures in quantum information theory. Inadequacy of the additive Shannon von-Neumann entropy as a measure of the information content of a quantum system has been pointed out [22]. Also, there is a theoretical observation [23] that the measure of quantum entanglement may not be additive. Despite the fact that the basic idea of quantum entanglement was acknowledged almost as soon as quantum theory was discovered, it is only in the last few years, that consideration has been given to finding mathematical methods to generally quantify entanglement.

In the case of a pure quantum state of two subsystems, a number of widely accepted measures of entanglement are known. However, the question of quantifying the degree of entanglement for general mixed states is still under discussion. Let us now briefly repeat some of the key underlying definitions. The entropy S of a quantum-mechanical system described by the density operator $\hat{\rho}$ is defined as follows:

$$S = -Tr\{\hat{\rho} \ln \hat{\rho}\}, \tag{15}$$

where we have set the Boltzmann constant K equal to unity. If $\hat{\rho}$ describes a pure state, then $S = 0$, and if $\hat{\rho}$ describes a mixed state, then $S \neq 0$. Entropies of the atomic and field sub-systems are defined by the corresponding reduced density operators:

$$S_{a(f)} = -Tr_{a(f)}\{\hat{\rho}_{a(f)} ln \rho_{a(f)}\}. \tag{16}$$

Taking the partial trace over the field, the reduced atomic matrix can be written as

$$
\begin{aligned}
\rho_t^a &= Tr_f(\rho) = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{n=0}^{\infty} |A_n(t)|^2 & \sum\limits_{n=0}^{\infty} A_n(t) B_{n+k}^*(t) \\ \sum\limits_{n=0}^{\infty} B_{n+k}(t) A_n^*(t) & \sum\limits_{n=0}^{\infty} |B_{n+k}(t)|^2 \end{pmatrix}.
\end{aligned}
\tag{17}
$$

Thus we rigorously obtain the quantum atomic entropy in the following form

$$S(\rho_t^a) = -\lambda_+^a(t) \log \lambda_+^a(t) - \lambda_-^a(t) \log \lambda_-^a(t), \tag{18}$$

where $\lambda_i^a(t)$ is given by

$$\lambda_\pm^a(t) = \frac{1}{2}\left\{1 \pm \sqrt{(2\rho_{ee}(t) - 1)^2 + 4|\rho_{eg}(t)|^2}\right\}. \tag{19}$$

In this case, the probability of finding the atom in its excited or ground states are expressed as the diagonal element of the reduced atomic density matrix, i.e.,

$$\rho_{ii}(t) = \langle i|\rho_t^a|i\rangle, \qquad i = e, g \tag{20}$$

and the off-diagonal element $\rho_{eg}(t)$ is given by

$$\rho_{ij}(t) = \langle i|\rho_t^a|j\rangle, \qquad i = e, g. \tag{21}$$

Taking the partial trace over the atomic system, we obtain the reduced density operator in the form

$$\rho_t^f = tr_A \rho(t), \tag{22}$$

with its $(\rho_t^f)_{nm}$ element given by

$$(\rho_t^f)_{nm} = A_n(t)A_m^*(t) + B_{n-k}(t)B_{m-k}^*(t). \tag{23}$$

From this equation, it is difficult to obtain the eigenvalues of the reduced density operator for the field, in this paper we will evaluate them numerically.

4 Results and conclusion

We study the temporal behavior of the atom-field system in the JC-model for the cavity-field prepared initially in different forms. As an example we may consider a simple initial condition for the atom to be in the excited state and the field in a coherent state or a superposition of the coherent state or in a statistical mixture of two coherent states (equations (3) and (4)). Among the family of mixed quantum mechanical states, special status should be accorded to those for a given value of the entropy and have the largest possible degree of entanglement. The reason for this is that such states can be regarded as mixed-state generalizations of Bell states, the latter being known to be the maximally entangled two-qubit pure states. Hence, this kind of mixed states could be expected to provide useful resources for quantum information processing. At this end, we have the plot of the quantum partial entropies (S_a, S_f) relative to these different initial states of the atom-field, as a function of the scaled time $\lambda t/\pi$, taking into account the two-photon process ($k = 2$), in order to investigate the Stark shift effects.

We assume a fixed value of the initial mean number of quanta $\bar{n} = 16$ and different values of Stark shift parameter R (namely, $\beta_1 = \beta_2 = 0$, i.e. in the absence of Stark shift in figure 1, $R = 0.5$ in figure 2 and $R = 0.3$ for figure 3). Furthermore, the detuning parameter is taken to be zero, and in figure $1a$ we set $r = 0$ (coherent state), figure $1b$, we set $r = 1$ (even coherent state) and figure $1c$ (a statistical mixture state). In the absence of the Stark shift, it is observed that the quantum field entropy and the quantum atomic entropy have the same values due to the initial coherent state $r = 0$, (see figure 1a). This behavior is similar to that obtained in the standard two-photon two-level systems obtained previously (see for example [4-5]). It is observed that the entropy evolves with a period π/λ, when $t = n\pi/\lambda, n = 0, 1, 2, ...$), the quantum field entropy evolves to zero and the field is completely disentangled from the atom, while for $t = (n+\frac{1}{2})\pi/\lambda$, it evolves to the maximum value, and the field is strongly entangled with the atom.

The situation is completely changed when we consider an even coherent state i.e $r = 1$.
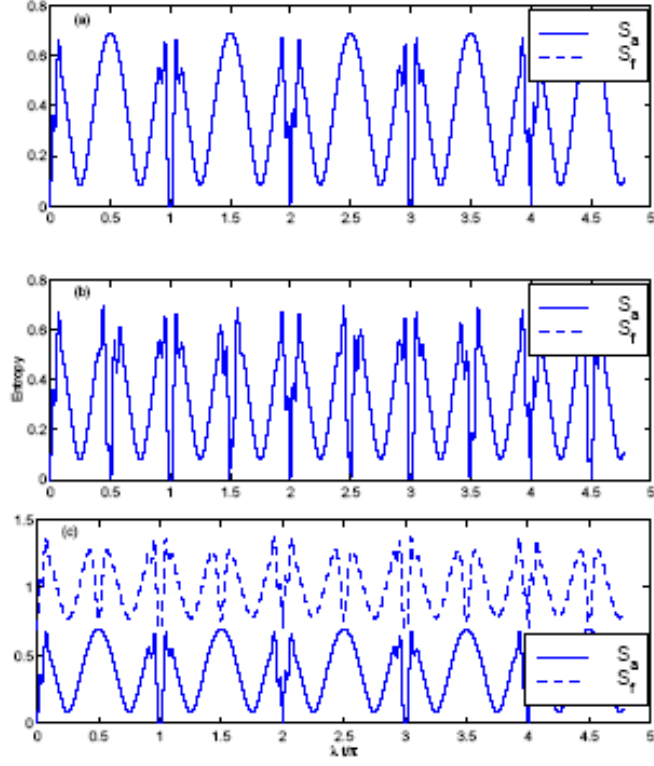
Figure 1: The entropy for the atom $S_a(t)$ (solid line) and the field entropy $S_f(t)$ (dashed line) as a function of the scaled time $\lambda t/\pi$ of the particle initially prepared the excited state and the field initially prepared in: (a) a superposition state SS ($r = 0$ (coherent state)), (b) a superposition state ($r = 1$ (even coherent state)) and (c) a statistical mixture (MS) of coherent states $\mid \alpha \rangle$ and $\mid -\alpha \rangle$ ($\bar{n} = 16$).

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Although the quantum field entropy is still equal the atomic entropy, the quantum entropy in this case has minimum values at $\lambda t = \frac{\pi n}{2}$, $(n = 0, 1, 2, ...)$ i.e at half of the revival time, also instead of the two pre-minimum values observed in figure 1a we have here only one pre-minimum value between each two consecutive minima. This effect is due to the interference between the two coherent states in the superposition, and can be understood by looking at the photon number distribution of the initial field. Hence this signal gives a clear measure of the remaining degree of coherence between the two components of the Schrodinger cat state, while the signals present in both cases are due to intrinsic revivals of each component individually. Because of this enhancement it is possible to have the generation of well-defined Schrödinger cat-like states during the evolution of the field in the two-photon process case model [24]. We would like to remark that the approach to a pure state at half of the revival time occurs in the ordinary JC-model [25], if we start with the field in a pure state. Let us now come to specific numerical examples to investigate the
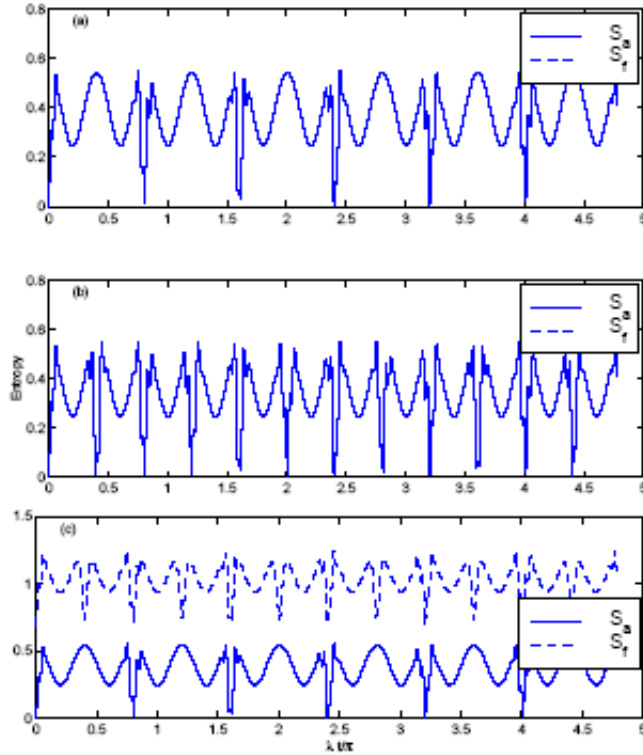

Figure 2: The same as in figure 1 but with the Stark shift parameter $R = 0.5$

influence of the statistical mixture on the evolution of the quantum field entropy and the quantum atomic entropy (see figure 1c). An understanding of interaction between an atom

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and an electromagnetic field has been possible in recent years through the introduction of the statistical mixture state picture [26]. By looking to the statistical mixture state one has a clear physical understanding of what are the parameters involved in such expression and what is going to neglect in order to go from a pure state to a mixed state. On the other words, the state of the initial field is an equally-weighted statistical mixture of two coherent states, which is a special class of the Schrodinger cat state. In this case, as it has been already discussed [20], the quantum field entropy $S_f(t)$ is greater than the quantum atomic entropy $S_a(t)$ (see figure 1c), $S_a(t)$ reach its maximum values at half of the revival time, while $S_f(t)$ evolves to minimum values. We note a little deviation from ordinary Rabi oscillations, due to the statistical mixture case. It is seen that entanglement evolves depending on the initial preparation of atoms, however from figures 1a and 1c we see that the quantum atomic entropy has similar behavior in both superposition and statistical mixture of coherent state.
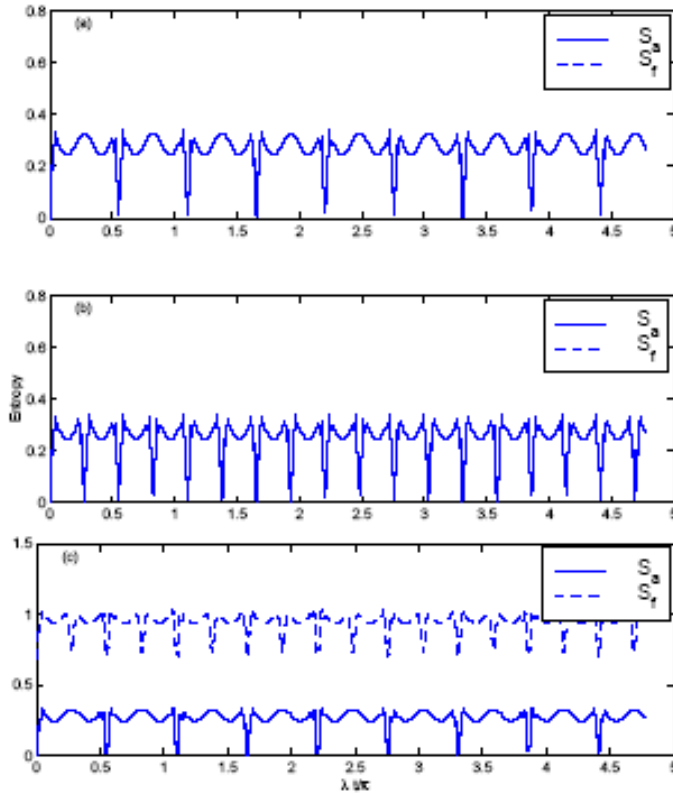

Figure 3: The same as in figure 1 but with the Stark shift parameter $R = 0.3$

For comparison purposes, we have chosen to set some different values of the Stark shift parameter R and the other parameters are the same as in figure 1. The outcome is presented in figure 2 (where $R = 0.5$) and figure 3 (where $R = 0.3$). We note a stronger modulation in the oscillations and a clear departure from ordinary Rabi oscillations being verified as the Stark shift parameter takes values far from the unity (see figures 2 and 3). Further studies about such a situation have been carried out and are not displayed here. It is interesting to refer here to the fact that the Stark shift creates an effective intensity dependent detuning $\Delta_N = \beta_2 - \beta_1$ [27]. When $\beta_2 = \beta_1, \quad \Delta_N = 0$, in this case, the Stark shift does not affect the time evolution of the quantum entropy. As is visible from the figures, the effects of the dynamic Stark shift are more pronounced when R deviates from unity. Interestingly, when R is decreases, the values of the maximum entropy are decreased. Periodic models therefore may be more robust in this sense. Also, with decreasing the parameter R, the evolution period of the entropy as well as the subentropies is decreases (see figure 3). The sensitivity becomes even more clearly visible when we take small velues of the Strak shift parameter. It is worth mentioning that the Strak shift effect has the same impact for both the field entropy and the atomic entropy.

When the atomic entropy is calculated, we note that the terms involved are of the forms $\langle U(t)\alpha | U(t)\alpha \rangle$, $\langle U(t)\alpha | - U(t)\alpha \rangle$ and $| \langle U(t)\alpha | U(t)\alpha \rangle |^2$ for the case of the mixture. These terms in particular do not differs from those of the case of the coherent state which has been discussed earlier. Therefore the temporal evolution of the entropy for the atomic system alone in the case of the Schrödinger cat sate (S) mimics the evolution of the entropy in the pure coherent state as can be seen from comparing figures (1c) and (1a). However when we consider the entropy for the field we note that terms of the form $\langle \pm U(t)\alpha | \pm U(t)\alpha \rangle$ with all combinations. These terms are the ones that appear in the case of the superposition of the two states ($r = 1$ in equation 4). Hence the resemblance between the figures for the entropy of the field in the mixed state of figure (1c) and the case of the initial superposed states of figure (1b). This may demonstrate relevance of investigating the entropies of the subsystems and their relation to entanglement.

In summary, we have shown in this paper that the final analytical expression of the composite density matrix along with its overlap matrix elements can be used to obtain the quantum field entropy and the quantum atomic entropy. This is accomplished by choosing to study the system in the representation in which the marginal initial density matrices are assumed to be in a coherent state, superposition states and statistical mixture states of two coherent states. We present different numerical examples to elucidate the

effects of these different settings. Explicit computations are presented for different values of the Stark shift parameter. Our results show that the superposition of coherent states and Stark shift play an important role in the evolution of the quantum entropies in the two-quanta JC-model. In the coherent state, the quantum field (atom) entropy reaches its maximum values at the half of the revival time and the field and the atom are strongly entangled, while in the even coherent state, the entropies at the same time evolve to the minimum values (zero) and the field and the atom are strongly disentangled. In the statistical mixture state, $(S_a \neq S_f)$, the entropy for the atom reach to the maximum values at half of the revival time, while S_f evolves to its minimum values. The significant effect of the Stark shift parameter appears when R deviates from unity. The more R deviates the more the two systems are weakly entangled.

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